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ON COEFFICIENT ESTIMATES FOR NEW SUBCLASSES OF q-BI-UNIVALENT FUNCTIONS

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ABSTRACT

In this paper, we introduce and investigate two new subclasses of the function class Σ of λ-q-bi-spirallike functions defined in the open unit disc. Furthermore, We find estimates on the coefficients $|a_2|, |a_3|$ and $|a_4|$ for functions in these new subclasses .

KEYWORDS:

Univalent functions,
Bi-univalent functions,
q-λ-spirallike,
Coefficients bounds.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open disc $E = \{z: z \in \mathbb{C}\ and\ |z| < 1\}$. Let S denote the subclass of function in $\mathcal{A}$ which are univalent in E and indeed normalized by $f(0) = f'(0) - 1 = 0$. It is well known that every function $f \in S$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \ \ (z \in E),$$

and

$$f\left(f^{-1}(\omega)\right) = \omega, \left(|\omega| < r_0(f), r_0(f) \geq \frac{1}{4}\right).$$

A function $f \in \mathcal{A}$ is said to bi-univalent function in E if f and f^{-1} are together univalent functions in E. Let Σ denote the class of bi-univalent functions defined in E. The inverse function $f^{-1}(\omega)$ is given by

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_3^2 - 5a_2a_3 + a_4)\omega^4 + \ldots. \quad (2)$$

A function ϕ is subordinate to a function φ, written as follows: $\phi(z) \prec \varphi(z), (z \in E)$, if there exists $\omega(z)$ analytic function in E such that $\omega(0) = 0$,and $\phi(z) = \varphi(\omega(z))$, $(|\omega(z)| < 1, z \in E$.

Let $\mathcal{S}_\Sigma^*(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ denote the classes Ma-Minda bi-starlike and bi-convex in E respectively. In the sequel, it is assumed that ϕ is an analytic function with positive real part in E such that $\phi(0) = 1, \phi'(0) > 0$ and $\phi(E)$ is symmetric with respect to the real axis. Such a function has a series expansion of the following from:

$$\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots, (c_1 > 0,\ z \in E).$$

We recall here a general Hurwite-Lerch Zeta function $\psi(z, s, a)$ defined in [7] by is given by

$$\psi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}.$$

Now we recall the definition of generalized Hurwitz-Lerch zeta function and a linear operator due to Ibrahim and Darus [10] as below:

$$\Theta_n(z, s, a) = \frac{\psi(z,s,a+nv)}{\psi(z,s,a)}, n \in N \cup \{0\}. \quad (3)$$

it is clear that $\Theta_0(z, s, a) = 1$. Further considering the function

$$z\Upsilon_\mu(z, s, a) = z + \sum_{n=2}^{\infty} \frac{\mu_{n-1}}{(n-1)} \Theta_{n-1}(z, s, a) z^n.$$

Ibrahim and Darus [10] defined the linear operator $(\Upsilon_\mu(z, s, a))^{-1} * f(z) = I_\mu^\delta(z, s, a): \mathcal{A} \longrightarrow \mathcal{A}$ and is given by

$$I_\mu^\delta(z, s, a) f(z) = \mathcal{J}_\mu^\delta f(z) = z + \sum_{n=2}^{\infty} \Psi_n a_n z^n,$$

Where $\Psi_n = \frac{\delta_{n-1}}{\mu_{n-1}\Theta_{n-1}(z,s,a)}, \mu \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}, a \in \mathbb{C}\backslash\{-(m+vn)\}, v \in \mathbb{C}\backslash\{0\}, n,m \in N \cup \{0\}, |s| < 1, |z| < 1$, and $\Theta_n(z,s,a)$ is defined in (3) and evidently we have

$$\Psi_2 = \frac{\delta_1}{\mu_1\Theta_1(z,s,a)} \; and \; \Psi_3 = \frac{\delta_2}{\mu_2\Theta_2(z,s,a)}. \tag{4}$$

In this paper we introduce two new subclasses of bi-univalent function class Σ by making use of the operator $\mathcal{J}_\mu^\delta f(z)$ and obtain estimates of the coefficients $|a_2|$ and $|a_3|$. Further Fekete-Szegö inequalities for the function class are determined. We now have the following definitions:

Definition 1.1. *The function $f(z)$, given by (1), is said to be a member of λ-SP_Σ^β the class of strongly λ-bi-spirallike functions of order $\beta, (|\lambda| \le \pi/2, 0 \le \beta < 1)$, if each of the following conditions are satisfied:*

$$f \in \Sigma \; and \left|arg\left(e^{i\lambda}\frac{zf'(z)}{f(z)}\right)\right| < \beta/2, (z \in E) \tag{5}$$

and

$$\left|arg\left(e^{i\lambda}\frac{\omega g'(\omega)}{g(\omega)}\right)\right| < \beta/2, (\omega \in E). \tag{6}$$

In [11], Jackson introduced and studied the concept of the q-derivative operator ∂_q as follows :

$$\partial_q f(z) = \frac{f(z)-f(qz)}{z(1-q)}, \; (z \neq 0, 0 < q < 1, \; \partial_q f(0) = f'(0)). \tag{7}$$

Equivalently(7), may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \; z \neq 0, \tag{8}$$

where $[n]_q = \frac{1-q^n}{1-q}$, note that as $q \to 1^-, [n]_q \to n$.

Definition 1.2. *Let $h(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$ be an univalent function in E such that $h(0) = 1, \Re(h(z)) > 0$. The function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{M}_\Sigma^{\mu,\delta}(\beta,\lambda,h,q)$, if it satisfies the following conditions:*

$$e^{i\lambda}\frac{z\partial_q(\mathcal{J}_\mu^\delta f(z))}{(1-\beta)z+\beta\mathcal{J}_\mu^\delta f(z)} \prec h(z)\cos\lambda + i\sin\lambda \tag{9}$$

and

$$e^{i\lambda}\frac{z\partial_q(\mathcal{J}_\mu^\delta g(\omega))}{(1-\beta)\omega+\lambda\mathcal{J}_\mu^\delta g(\omega)} \prec h(\omega)\cos\lambda + i\sin\lambda, \tag{10}$$

where $\lambda \in (-\pi/2, \pi/2), 0 \le \beta \le 1$, the function g is given by (2) and $z, \omega \in E$.

Definition 1.3. *Let*

$$h(z) = \frac{1-z}{2(\beta^2-[2]_q\beta)\Phi_2^2+2([3]_q-\beta)\Phi_3}, \tag{11}$$

be an univalent function in E such that $h(0) = 1$, $\Re(h(z)) > 0$. The function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{K}_\Sigma^{\mu,\delta}(\beta, \lambda, h, q)$, if it satisfies the following conditions:

$$e^{i\lambda}\frac{z\partial_q(\mathcal{J}_\mu^\delta f(z))+z^2\partial_q^2(\mathcal{J}_\mu^\delta f(z))}{(1-\beta)z+\beta z\,\partial_q(\mathcal{J}_\mu^\delta f(z))} \prec h(z)\cos\lambda + i\sin\lambda \tag{12}$$

and

$$e^{i\lambda}\frac{\omega\partial_q(\mathcal{J}_\mu^\delta g(\omega))+\omega^2\partial_q^2(\mathcal{J}_\mu^\delta g(\omega))}{(1-\beta)\omega+\beta\omega\,\partial_q(\mathcal{J}_\mu^\delta g(\omega))} \prec h(\omega)\cos\lambda + i\sin\lambda, \tag{13}$$

where $\lambda \in (-\pi/2, \pi/2), 0 \le \beta \le 1$, the function g is given by (2) and $z, \omega \in E$.

Remark 1.1. *If the function $h(z) = \frac{1+Az}{1+Bz}$ the class $\mathcal{M}_\Sigma^{\mu,\delta}(\lambda, \beta, h, q) \equiv \mathcal{M}_\Sigma^{\mu,\delta}(\lambda, \beta, A, B, q)$ and satisfies the following conditions:*

$$e^{i\lambda}\frac{z\partial_q(\mathcal{J}_\mu^\delta f(z))}{(1-\beta)z+\beta\mathcal{J}_\mu^\delta f(z)} \prec \frac{1+Az}{1+Bz}\cos\lambda + i\sin\lambda \tag{14}$$

and

$$e^{i\lambda}\frac{\omega\partial_q(\mathcal{J}_\mu^\delta g(\omega))}{(1-\beta)\omega+\beta\mathcal{J}_\mu^\delta g(\omega)} \prec \frac{1+A\omega}{1+B\omega}\cos\lambda + i\sin\lambda, \tag{15}$$

where $\lambda \in (-\pi/2, \pi/2), 0 \le \beta \le 1, -1 \le B < A \le 1$, the function g is given by (2) and $z, \omega \in E$.

Remark 1.2. *If the function $h(z) = \frac{1+(1-2\alpha)z}{1-z}$ the class $\mathcal{M}_\Sigma^{\mu,\delta}(\lambda, \beta, h, q) \equiv \mathcal{M}_\Sigma^{\mu,\delta}(\lambda, \beta, \alpha, q)$ and satisfies the following conditions:*

$$\Re\left(e^{i\lambda}\frac{z\partial_q(\mathcal{J}_\mu^\delta f(z))}{(1-\beta)z+\beta\mathcal{J}_\mu^\delta f(z)}\right) > \alpha\cos\lambda \tag{16}$$

and

$$\Re\left(e^{i\lambda}\ \frac{\omega\partial_q(\mathcal{J}_\mu^\delta g(\omega))}{(1-\beta)\omega+\beta\mathcal{J}_\mu^\delta g(\omega)}\right) > \alpha\cos\lambda, \tag{17}$$

where $\lambda\in(-\pi/2,\pi/2), 0\le\beta\le1, -1\le\alpha<1$, the function g is given by (2) and $z,\omega\in E$.

If i taken $h(z)=\frac{1+Az}{1+Bz}$ or $h(z)=\frac{1+(1-2\alpha)z}{1-z}$, we state analogous subclasses of $\mathcal{K}_\Sigma^{\mu,\delta}(\lambda,\beta,h,q)$ as in above remark 1.1, and 1.2, of respectively.

We need this the following lemma:

Lemma 1.1. *[18]Let $\phi(z)$ given by $\phi(z)=\sum_{n=1}^{\infty}B_n z^n, (z\in E)$ be convex in E. Suppose that $h(z)=\sum_{n=1}^{\infty}h_n z^n$, is holomorphic in E. If $h(z)\prec\phi(z), (z\in E)$ then $|h(z)|\le|B_1|, (n\in N)$.*

Lemma 1.2. *[17]If $p\in\mathcal{P}$ then $|p_k|\le 2, (k\in N)$ where $\mathcal{P}$is the family of all functions p analytic in E which $\Re(p(z))>0, (z\in E)$,where $p(z)=1+p_1z+p_2z^2+p_3z^3+\ldots, z\in E$.*

2 *MAIN RESULTS*

Theorem 2.1. *Let $f\in\mathcal{A}$given by (1) in the class $\mathcal{M}_\Sigma^{\mu,\delta}(\lambda,\beta,h,q)$, then*

$$|a_2|\le\sqrt{\frac{|B_1|\cos\lambda}{(\beta^2-[2]_q\beta)\Psi_2^2+([3]_q-\beta)\Psi_2}}, \tag{18}$$

$$|a_3|\le\frac{|B_1|\cos\lambda}{([3]_q-\beta)\Psi_3}+\left(\frac{|B_1|\cos\lambda}{([2]_q-\beta)\Psi_2}\right)^2, \tag{19}$$

where $\lambda\in(-\pi/2,\pi/2), 0\le\beta\le1$, and Ψ_2,Ψ_3 are given by (4).

Proof. From (9) and (10) that

$$e^{i\lambda}\ \frac{z\,\partial_q(\mathcal{J}_\mu^\delta f(z))}{(1-\beta)z+\beta\mathcal{J}_\mu^\delta f(z)} = P(z)\cos\lambda + i\sin\lambda\ \ (z\in E), \tag{20}$$

$$e^{i\lambda}\ \frac{\omega\partial_q(\mathcal{J}_\mu^\delta g(\omega))}{(1-\beta)\omega+\beta\mathcal{J}_\mu^\delta g(\omega)} = q(\omega)\cos\lambda + i\sin\lambda\ \ (\omega\in E), \tag{21}$$

where $p(z)\prec h(z), (z\in E)$ and $q(\omega)\prec h(\omega), (\omega\in E)$, are have the following forms:

$$p(z)=1+c_1z+c_2z^2+c_3z^3+\ldots, z\in E,$$

$$q(\omega) = 1 + q_1(\omega) + q_2(\omega)^2 + q_3(\omega)^3 + \ldots, \omega \in E.$$

Now,

$$e^{i\lambda}\ \frac{z+[2]_q\Psi_2 a_2 z^2+[3]_q\Psi_3 a_3 z^3+\ldots}{z+\beta\Psi_2 a_2 z^2+\beta\Psi_3 a_3 z^3+\ldots} = P(z)\cos\lambda + i\sin\lambda \quad (z \in E), \tag{22}$$

and

$$e^{i\lambda}\ \frac{\omega+[2]_q\Psi_2 a_2 \omega^2+[3]_q\Psi_3 a_3 \omega^3+\ldots}{\omega+\beta\Psi_2 a_2 \omega^2+\beta\Psi_3 a_3 \omega^3+\ldots} = q(\omega)\cos\lambda + i\sin\lambda \quad (\omega \in E), \tag{23}$$

from (20) and (21), it follows that

$$e^{i\lambda}\ ([2]_q - \beta)\Psi_2 a_2 = c_1\cos\lambda, \tag{24}$$

$$e^{i\lambda}\ \left\{(\beta^2 - [2]_q\beta)\Psi_2^2 a_2^2 + ([3]_q - \beta)\Psi_3 a_3\right\} = c_2\cos\lambda, \tag{25}$$

$$-e^{i\lambda}\ ([2]_q - \beta)\Psi_2 a_2 = q_1\cos\lambda, \tag{26}$$

and

$$e^{i\lambda}\ \left\{(\beta^2 - [2]_q\beta)\Psi_2^2 a_2^2 + ([3]_q - \beta)(2a_2^2 - a_3)\Psi_3\right\} = q_2\cos\lambda. \tag{27}$$

From (24) and (26), we find that

$$c_1 = -q_1 \tag{28}$$

and

$$2e^{2i\lambda}\ ([2]_q - \beta)^2\Psi_2^2 a_2^2 = (c_1^2 + q_1^2)\cos^2\lambda, \tag{29}$$

then

$$a_2^2 = \frac{(c_1^2+q_1^2)\cos^2\lambda\, e^{-2i\lambda}}{2([2]_q-\beta)^2\Psi_2^2}. \tag{30}$$

Adding (25) and (27), we have

$$a_2^2 = \frac{(c_2+q_2)\cos\lambda\, e^{-i\lambda}}{2(\beta^2-[2]_q\beta)\Psi_2^2+([3]_q-\beta)\Psi_2}. \tag{31}$$

By applying Lemma 1.1 and 1.2, for the coefficients c_2 and q_2, we have $|c_k| = \frac{c^k(0)}{K} \leq |B_1|, (k \in N), |q_k| = \frac{q^k(0)}{K} \leq |B_1|, (k \in N)$ and using these in (31), we get

$$|a_2|^2 \leq \frac{(|c_2|+|q_2|)\cos\lambda}{2(\beta^2-[2]_q\beta)\Psi_2^2+([3]_q-\beta)\Psi_2} \leq \frac{|B_1|\cos\lambda}{2(\beta^2-[2]_q\beta)\Psi_2^2+([3]_q-\beta)\Psi_2}. \tag{32}$$

Now

$$|a_2| \leq \sqrt{\frac{|B_1|\cos\lambda}{(\beta^2-[2]_q\beta)\Psi_2^2+([3]_q-\beta)\Psi_2}}. \tag{33}$$

From (25) and (27), we get

$$a_3 - a_2^2 = \frac{(c_2 - q_2)\cos\lambda\, e^{-i\lambda}}{2([3]_q - \beta)\Psi_3}. \tag{34}$$

Substituting value of a_2^2 from (30) and (34), we get

$$a_3 = \frac{(c_2 - q_2)\cos\lambda\, e^{-i\lambda}}{2([3]_q - \beta)\Psi_3} + \frac{(c_1^2 + q_1^2)\cos^2\lambda\, e^{-2i\lambda}}{2([2]_q - \beta)^2\Psi_2^2}. \tag{35}$$

Also applying Lemma 1.1 and 1.2, for the coefficients c_2 and q_2, we have

$$|a_3| \leq \frac{|B_1|\cos\lambda}{([3]_q - \beta)\Psi_3} + \left(\frac{|B_1|\cos\lambda}{([2]_q - \beta)\Psi_2}\right)^2. \tag{36}$$

As $q \to 1^-$ in the above Theorem we get the following result proved by Janani [12].

Corollary 2.1.*Let $f \in \mathcal{A}$given by (1) in the class $\mathcal{M}_\Sigma^{\mu,\delta}(\lambda, \beta, h)$, then*

$$|a_2| \leq \sqrt{\frac{|B_1|\cos\lambda}{(\beta^2 - 2\beta)\Psi_2^2 + (3 - \beta)\Psi_2}}, \tag{37}$$

$$|a_3| \leq \frac{|B_1|\cos\lambda}{(3 - \beta)\Psi_3} + \left(\frac{|B_1|\cos\lambda}{(2 - \beta)\Psi_2}\right)^2. \tag{38}$$

As $q \to 1^-$ and $h(z) = \frac{1+Az}{1+Bz}, (-1 \leq B < A \leq 1)$, we get the following result proved by Janani [12].

Corollary 2.2.*Let $f \in \mathcal{A}$given by (1) in the class $\mathcal{M}_\Sigma^{\mu,\delta}(\lambda, \beta, A, B)$, t hen*

$$|a_2| \leq \sqrt{\frac{(A - B)\cos\lambda,}{(\beta^2 - 2\beta)\Psi_2^2 + (3 - \beta)\Psi_2}}, \tag{39}$$

$$|a_3| \leq \frac{(A - B)\cos\lambda}{(3 - \beta)\Psi_3} + \left(\frac{(A - B)\cos\lambda}{(2 - \beta)\Psi_2}\right)^2, \tag{40}$$

where $\lambda \in (-\pi/2, \pi/2), 0 \leq \beta \leq 1$ and Ψ_2, Ψ_3 are given by (4).

As $q \to 1^-$ and $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, we get the following result proved by Janani [12].

Corollary 2.3. *Let $f \in \mathcal{A}$given by (1) in the class $\mathcal{M}_\Sigma^{\mu,\delta}(\lambda, \beta, \alpha, q)$, then*

$$|a_2| \leq \sqrt{\frac{2(1 - \alpha)\cos\lambda}{(\beta^2 - [2]_q\beta)\Psi_2^2 + ([3]_q - \beta)\Psi_2}}, \tag{41}$$

$$|a_3| \leq \frac{2(1 - \alpha)\cos\lambda}{([3]_q - \beta)\Psi_3} + \left(\frac{2(1 - \alpha)\cos\lambda}{([2]_q - \beta)\Psi_2}\right)^2, \tag{42}$$

where $\lambda \in (-\pi/2, \pi/2), 0 \leq \beta \leq 1$, and Ψ_2, Ψ_3 are given by (4)

Theorem 2.2. *Let $f \in \mathcal{A}$ given by (1) in the class $\mathcal{K}_{\Sigma}^{\mu,\delta}(\lambda, \beta, h, q)$, then*

$$|a_2| \leq \sqrt{\frac{|B_1|\cos^2\lambda}{\left\{[2]_q^2(\beta^2-[2]_q\beta)\Psi_2^2+[3]_q(([2]_q+1)-\beta)\Psi_2\right\}}}, \quad (43)$$

$$|a_3| \leq \frac{|B_1|\cos\lambda}{[3]_q(([2]_q+1)-\beta)\Psi_3} + \left(\frac{|B_1|\cos\lambda}{[2]_q([2]_q-\beta)\Psi_2}\right)^2, \quad (44)$$

where $\lambda \in (-\pi/2, \pi/2), 0 \leq \beta \leq 1$, and Ψ_2, Ψ_3 are given by (4).

Proof. From (20) and (21), we get

$$e^{i\lambda}\ \frac{z+[2]_q\Psi_2 a_2 z^2+[3]_q\Psi_3 a_3 z^3+\ldots+[2]_q\Psi_2 a_2 z^2+[2]_q[3]_q\Psi_3 a_3 z^3+\ldots}{z+[2]_q\Psi_2 a_2\beta z^2+[3]_q\Psi_3 a_3\beta z^3+\ldots} = p(z)\cos\lambda + i\sin\lambda.$$

(45)

Then

$$[2]_q e^{i\lambda}\ (2-\beta)\Psi_2 a_2 = c_1\cos\lambda, \quad (46)$$

$$e^{i\lambda}\ \left\{[2]_q^2(\beta^2-[2]_q\beta)\Psi_2^2 a_2^2 + [3]_q(([2]_q+1)-\beta)\Psi_3 a_3\right\} = c_2\cos\lambda, \quad (47)$$

$$-[2]_q e^{i\lambda}\ (2-\beta)\Psi_2 a_2 = q_1\cos\lambda \quad (48)$$

and

$$\left\{[2]_q^2(\beta^2-[2]_q\beta)\Psi_2^2 a_2^2 + [3]_q(([2]_q+1)-\beta)(2a_2^2-a_3)\Psi_3\right\} = q_2\cos\lambda. \quad (49)$$

From (46) and (48), we get

$$c_1 = -q_1, \quad (50)$$

$$2[2]_q^2 e^{2i\lambda}\ (2-\beta)^2\Psi_2^2 a_2 = (c_1^2+q_1^2)\cos^2\lambda \quad (51)$$

$$a_2^2 = \frac{(c_1^2+q_1^2 2)\cos^2\lambda\, e^{-2i\lambda}}{2[2]_q^2(2-\beta)^2\Psi_2^2}. \quad (52)$$

Adding (47) and (49), we get

$$a_2^2 = \frac{(c_2+q_2)\cos^2\lambda\, e^{-i\lambda}}{2\left\{[2]_q^2(\beta^2-[2]_q\beta)\Psi_2^2+[3]_q(([2]_q+1)-\beta)\Psi_2\right\}}, \quad (53)$$

applying Lemma 1.1 and 1.2, for the coefficients c_2 and q_2, we have

$$|a_2|^2 \leq \frac{(|c_2|+|q_2|)\cos^2\lambda}{2\left\{[2]_q^2(\beta^2-[2]_q\beta)\Psi_2^2+[3]_q(([2]_q+1)-\beta)\Psi_2\right\}}. \quad (54)$$

Now

$$|a_2| \leq \sqrt{\frac{|B_1|\cos^2\lambda\, e^{-i\lambda}}{\{[2]_q^2(\beta^2-[2]_q\beta)\Psi_2^2+[3]_q(([2]_q+1)-\beta)\Psi_2\}}}. \tag{55}$$

From (47) and (49)

$$a_3 - a_2^2 = \frac{(c_2-q_2)\cos^2\lambda\, e^{-i\lambda}}{[3]_q(([2]_q+1)-\beta)\Psi_3}, \tag{56}$$

Substituting value of a_2^2 from (52) and (56), we get

$$a_3 = \frac{(c_2-q_2)\cos^2\lambda\, e^{-i\lambda}}{[3]_q(([2]_q+1)-\beta)\Psi_3} + \frac{(c_1^2+q_1^2 2)\cos^2\lambda\, e^{-2i\lambda}}{2[2]_q^2(2-\beta)^2\Psi_2^2}. \tag{57}$$

Also applying Lemma 1.1 and 1.2, for the coefficients c_2 and q_2, we have

$$|a_3| \leq \frac{|B_1|\cos\lambda}{[3]_q(([2]_q+1)-\beta)\Psi_3} + \left(\frac{|B_1|\cos\lambda}{[2]_q(2-\beta)\Psi_2}\right)^2. \tag{58}$$

As $q \to 1^-$ in the above Theorem we get the following result proved by Janani [12].

Corollary 2.4. *Let $f \in \mathcal{A}$ given by (1) in the class $\mathcal{H}_\Sigma^{\mu,\delta}(\lambda,\beta,h)$, then*

$$|a_2| \leq \sqrt{\frac{|B_1|\cos\lambda}{4(\beta^2-2\beta)\Psi_2+3(3-\beta)\Psi_3}}, \tag{59}$$

$$|a_3| \leq \frac{|B_1|\cos\lambda}{3(3-\beta)\Psi_3} + \left(\frac{|B_1|\cos\lambda}{2(2-\beta)^2\Psi_2^2}\right)^2. \tag{60}$$

Theorem 2.3. *Let $f \in \mathcal{A}$ given by (1) in the class $\mathcal{M}_\Sigma^{\mu,\delta}(\lambda,\beta,\alpha,q)$, then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2\cos\lambda\, B_1|h(\eta)|, & |h(\eta)| > \frac{1}{2([3]_q-\beta)\Psi_3}, \\ \frac{B_1\cos\lambda}{([3]_q-\beta)\Psi_3}, & |h(\eta)| < \frac{1}{2([3]_q-\beta)\Psi_3}, \end{cases} \tag{61}$$

where

$$h(\eta) = \frac{1-\eta}{2(\beta^2-[2]_q\beta)\Psi_2^2+2([3]_q-\beta)\Psi_3}. \tag{62}$$

Proof. From (34), we have

$$a_3 = a_2^2 + \frac{(c_2-q_2)\cos\lambda\, e^{-i\lambda}}{2([3]_q-\lambda)\Psi_3}. \tag{63}$$

We compensate for the value of $a - 2^2$ given by (33) in (34), we get

$$a_3 - \eta a_2^2 = e^{-i\lambda} \cos\lambda \left[\left(h(\eta) + \frac{1}{2([3]_q - \beta)\Psi_3}\right) c_2 + \left(h(\eta) - \frac{1}{2([3]_q - \beta)\Psi_3}\right) q_2\right],$$

where

$$h(\eta) = \frac{1-\eta}{2(\beta^2-[2]_q\beta)\Psi_2^2+2([3]_q-\beta)\Psi_3}.$$

As $q \to 1^-$ in the above Theorem we get the following result proved by Janani [12].

Corollary 2.5.

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2\cos\lambda\, B_1|h(\eta)|, & |h(\eta)| > \frac{1}{2(3-\beta)\Psi_3}, \\ \frac{B_1\cos\lambda}{(3-\beta)\Psi_3}, & |h(\eta)| < \frac{1}{2(3-\beta)\Psi_3}, \end{cases} \quad (64)$$

where

$$h(\eta) = \frac{1-\eta}{2(\beta^2-2\beta)\Psi_2^2+2(3-)\Psi_3}. \quad (65)$$

Theorem 2.4. *Let the function f given by (1) in the class $\mathcal{K}_\Sigma^{\mu,\delta}(\lambda,\beta,\alpha,q)$, then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2\cos\lambda\, B_1|h_\beta(\eta)|, & |h(\eta)| > \frac{1}{2[3]_q([3]_q-\beta)\Psi_3}, \\ \frac{B_1\cos\lambda}{[3]_q([3]_q-\beta)\Psi_3}, & |h(\eta)| < \frac{1}{2[3]_q([3]_q-\beta)\Psi_3}, \end{cases} \quad (66)$$

where

$$h(\eta) = \frac{1-\eta}{2[2]_q^2(\lambda^2-[2]_q\lambda)\Psi_2^2+2[3]_q([3]_q-\lambda)\Psi_3}. \quad (67)$$

As $q \to 1^-$ in the above Theorem we get the following result proved by Janani [12].

Corollary 2.6. *Let $f \in \mathcal{A}$ given by (1) in the class $\mathcal{K}_\Sigma^{\mu,\delta}(\beta,\lambda,\alpha)$, then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2\cos\lambda\, B_1|h_\beta(\eta)|, & |h(\eta)| > \frac{1}{6(3-\beta)\Psi_3}, \\ \frac{B_1\cos\beta}{3(3-\lambda)\Psi_3}, & |h(\eta)| < \frac{1}{6(3-\lambda)\Psi_3}, \end{cases} \quad (68)$$

where

$$h(\eta) = \frac{1-\eta}{8(\beta^2-2\beta)\Psi_2^2+6(3-\beta)\Psi_3}. \quad (69)$$

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